

PRINCIPLES OF LOGARITHMS AND THE p OPERATOR

It is not too difficult to calculate whole positive powers of whole positive numbers eg.

$$2^2 = 4, 2^3 = 8, 3^2 = 9, 2^6 = 64, 10^6 = 1\ 000\ 000 \text{ etc.}$$

When the powers are not whole positive numbers it is a little more difficult. One really needs a scientific calculator with an " x^y " (or " y^x ") function. With this one simply enters the first number, presses the x^y button, enters the second number (the power) and then presses the *equals* button. The calculator immediately calculates the desired result. Try it yourself with the following and confirm the answers given:

$$4^{0.5} = 2$$

$$2^{0.5} = 1.414\dots$$

$$100^{0.1} = 1.58489\dots$$

$$100^{1.5} = 1000$$

$$6.1^{-3.2} = 0.003069\dots$$

$$0.1^{-0.1} = 1.258925\dots$$

$$-5.789^{-2} = 0.02984$$

If we call the three numbers in each case a, b and c, so that

$$a^b = c$$

then we can say that the *logarithm* of c to the *base* a is b, and we write

$$\log_a c = b \quad \text{(DEFINITION)}$$

That is to say, the *logarithm* of the number c to the *base* a, is the power to which a must be raised to get c.

Referring back to the first set of examples above we see that they lead directly to the following:

$$\log_2 4 = 2, \log_2 8 = 3, \log_3 9 = 2, \log_2 64 = 6, \log_{10} 1\ 000\ 000 = 6$$

and referring to the second set of examples:

$$\log_4 2 = 0.5, \log_2 1.414 = 0.5, \log_{-5.789} 0.02984 = -2$$

Note that one cannot take the *logarithm* of a negative number, although the *base* or the *result* may be negative.

The two most commonly used bases are 10, and "e", the so-called *natural number*. e is a number rather like π , in that it cannot be written down in a precise form. It has the value 2.718281828... (You can see

this on your calculator by using the e^x function on the number 1.) Logarithms to the base e are known as **natural logarithms** (or Napierian logarithms). They occur frequently in mathematics. Since these bases are so common, there are special short-hand methods of writing them:

$\log_{10}(x)$ is written $\lg(x)$

$\log_e(x)$ is written $\ln(x)$

Most calculators are not equipped to calculate logarithms to **any** base. Scientific calculators can generally calculate logarithms to the bases 10 and e , however. One simply enters the number and then presses the *log*, (or *lg*) button for \log_{10} , or the *ln* (sometimes wrongly written *In*) button for \log_e . Don't forget to pay careful attention to negative signs if present. If you try to take the logarithm of zero or a negative number you will get an error message.

To confirm your understanding of logarithms, try evaluating the following examples. This can be done from a consideration of the definition of a logarithm and by examining the previous examples carefully. You should not use a calculator except for checking your answers.

$\log_2 4$, $\log_3 9$, $\log_4 2$, $\log_2(\sqrt{2})$..., $\log_{100}(1.58489)$, $\log_{100}(1000)$, $\log_{6.1}(0.003069)$, $\log_{0.1}(1.258925)$, $\lg_{(-5.789)}(0.02984)$

Logarithms have some rather remarkable properties, which have been used in the past to allow them to be employed as aids to calculation. These can be stated generally in terms of two numbers "a" and "b". The logarithms could be to any base, so the expressions work equally well for logarithms to the base 10 as for natural logarithms.

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|------|---|----------------------------------|
| i) | $\log(a \times b) = \log(a) + \log(b)$ | logarithm of a product |
| ii) | $\log(a/b) = \log(a) - \log(b)$ | logarithm of a quotient |
| iii) | $\log(1/a) = -\log(a)$ | logarithm of a reciprocal |
| iv) | $\log(a^b) = b \times \log(a)$ | logarithm of a power |
| v) | $\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$ | changing the base of a logarithm |

For example, since $\lg(2) = 0.3010$ and $\lg(100) = 2$
 so $\lg(200) = \lg(2 \times 100) = \lg(2) + \lg(100) = 0.3010 + 2 = 2.3010$.

Similarly

$$\lg(50) = \lg(100/2) = \lg(100) - \lg(2) = 2 - 0.3010 = 1.6990$$

Try the following yourself given also that $\lg(3) = 0.4771$. Only use a calculator to check your answers.

- i) $\lg(2000) =$ ii) $\lg(500)$ iii) $\lg(6)$ iv) $\lg(0.6667)$ v) $\lg(1.5)$ vi) $\lg(1/3)$ vii) $\lg(33.333)$ viii) $\lg(8)$

ix) $\lg(27)$

The last two examples can be solved more simply by using the fourth property of those given, and the sixth example can be solved using the third property above.

The fifth property of logarithms as given above can be used for changing the base of a logarithm. Usually we wish to change between base 10 and base e. In this case the expression becomes:

$$\log_{10}(x) = \frac{\log_e(x)}{\log_e(10)}$$

Note how $\log_e(10)$ occurs. It is always the same whatever the value of x. In fact it turns out to be 2.303, as you can check with your calculator. Rearranging the above expression gives us

$$\log_e x = 2.303 \times \log_{10}(x)$$

Use this expression to find $\log_e(1)$, $\log_e(100)$, $\log_e(0.1)$, $\log_e(1000)$, $\log_e(0.01)$

THE p OPERATOR

This is closely related to log. In fact "p" means simply $-\log_{10}$. Thus $p10 = -1$, $p0.1 = +1$ etc. Simply find the \log_{10} and then change the sign. The rules of logarithms as given above work just as well with the "p" operator replacing the "log" operator.

- i) $p(a \times b) = p(a) + p(b)$
- ii) $p(a/b) = p(a) - p(b)$
- iii) $p(1/a) = -p(a)$
- iv) $p(a^b) = b \times p(a)$

Now try the following. You may need to use a calculator, but only for numbers (iii) and (vi).

- i) $p100$ ii) $p0.01$ iii) $p2$ iv) $p20$ v) $p5$ vi) $p(1/2)$ vii) $p(1/20)$

Now find the number whose "p" is each of the following. Note that if px is $-\text{Log}_{10}x$, then $x = 10^{-px}$.

- i) 2 ii) 3 iii) 0 iv) -1 v) -6 vi) -1/30 vii) -2/48